

On the number of nodal domains of the 2D isotropic quantum harmonic oscillator

– an extension of results of A. Stern –

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September 8, 2014

Abstract

In the case of the sphere and the square, Antonie Stern (1925) claimed in her PhD thesis the existence of an infinite sequence of eigenvalues whose corresponding eigenspaces contain an eigenfunction with two nodal domains. These two statements were given complete proofs respectively by Hans Lewy in 1977, and the authors in 2014 (see also Gauthier-Shalom-Przytykowski (2006)). The aim of this paper is to obtain a similar result in the case of the isotropic quantum harmonic oscillator in the two dimensional case.

Keywords: Quantum harmonic oscillator, Nodal lines, Nodal domains, Courant theorem.
MSC 2010: 35B05, 35Q40, 35P99, 58J50, 81Q05.

1 Introduction and main results

The aim of this paper is to construct a sequence of eigenvalues, and a corresponding sequence of eigenfunctions, for the 2D isotropic quantum harmonic oscillator

$$\widehat{H} := -\Delta + x^2 + y^2, \quad (1.1)$$

with exactly two nodal domains. Similar results were stated in Antonie Stern' PhD thesis (with R. Courant as advisor), for the Dirichlet problem for the Laplacian in the square,

and in the case of the Laplace-Beltrami operator on the sphere \mathbb{S}^2 [14]. This is only much later that H. Lewy [9] in the case of the sphere (see also [2]), and the authors [1] in the case of the square with Dirichlet conditions (see also Gauthier-Shalom-Przybytkowski [6]), have proposed complete proofs of her statements.

Coming back to the isotropic harmonic oscillator, an orthogonal basis of eigenfunctions is given by

$$\phi_{m,n}(x, y) = H_m(x)H_n(y) \exp\left(-\frac{x^2 + y^2}{2}\right), \quad (1.2)$$

for $(m, n) \in \mathbb{N}^2$, where $H_n(x)$ denotes the Hermite polynomial of degree n .

The eigenfunction $\phi_{m,n}$ corresponds to the eigenvalue $2(m + n + 1)$,

$$\hat{H}\phi_{m,n} = 2(m + n + 1)\phi_{m,n}. \quad (1.3)$$

Here we use the definitions and notation of Szegő [16, §5.5].

The eigenspace \mathcal{E}_ℓ associated with the eigenvalue $\hat{\lambda}(\ell) = 2(\ell + 1)$ has dimension $(\ell + 1)$, and is generated by the eigenfunctions $\phi_{\ell,0}, \phi_{\ell-1,1}, \dots, \phi_{0,\ell}$.

For $\theta \in [0, \pi]$, we shall consider the families of eigenfunctions,

$$\Phi_n^\theta := \cos \theta \phi_{n,0} + \sin \theta \phi_{0,n}, \quad (1.4)$$

corresponding to the eigenvalue $2(n + 1)$.

Our aim is to prove the following theorems.

Theorem 1.1 *Assume that n is odd. Then, there exists an open interval $I_{\frac{\pi}{4}}$ containing $\frac{\pi}{4}$, and an open interval $I_{\frac{3\pi}{4}}$, containing $\frac{3\pi}{4}$, such that for*

$$\theta \in I_{\frac{\pi}{4}} \cup I_{\frac{3\pi}{4}} \setminus \left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\},$$

the nodal set $N(\Phi_n^\theta)$ is a connected simple regular curve, and the eigenfunction Φ_n^θ has two nodal domains in \mathbb{R}^2 .

Theorem 1.2 *Assume that n is odd. Then, there exists $\theta_c > 0$ such that, for $0 < \theta < \theta_c$, the nodal set $N(\Phi_n^\theta)$ is a connected simple regular curve, and the eigenfunction Φ_n^θ has two nodal domains in \mathbb{R}^2 .*

As in the case of the square, to prove Theorem 1.1, we begin by a symmetry argument to reduce to a neighborhood of either $\pi/4$ or $3\pi/4$, say $3\pi/4$. The first step in the proof is to analyze the zero set when $\theta = \frac{3\pi}{4}$, in particular the points where the zero set is critical, and to show that this only occurs on the diagonal.

The second step is then to show that the double crossings on the diagonal disappear by perturbation, for θ close to and different from $\frac{3\pi}{4}$. Using the local nodal patterns and some barrier lemmas, one can then show that the nodal set becomes a connected simple curve, asymptotic to $x = y$ at $\pm\infty$. The local stability of the nodal set under perturbation, then gives an explicit interval containing $3\pi/4$ in which the phenomenon occurs (for $\theta \neq \frac{3\pi}{4}$ of course).

The proof of Theorem 1.2 follows similar lines.

Remarks.

1. In the case of the square the same kind of analysis is also interesting for determining when the number of nodal domains of an n -th eigenfunction is equal to n (the so called Courant sharp situation). Pleijel [11] observed that in the case of the square (Dirichlet condition) this only occurs for the first, the second and the fourth eigenfunctions (see [1] for a complete argument). In the case of the sphere, as a consequence of the analysis of Leydold [10], this only occurs for the first and the second eigenfunctions. It is natural to investigate a similar question in the case of the isotropic harmonic oscillator, and more generally, the validity of Pleijel's theorem in this case. In the last section, we will give a Leydold's like proof of the fact that the only Courant sharp eigenvalues of the harmonic oscillator are $\hat{\lambda}(\ell)$, for $\ell = 0, 1$ and 2. As communicated by I. Polterovich, this question will be analyzed from a different point of view in [3].
2. A connected question is to analyze the zero set when θ is a random variable. We refer to [7] for results in this direction.
3. These questions are related to the question of spectral minimal partitions [8]. In the case of the harmonic oscillator similar questions appear in the analysis of the properties of ultracold atoms (see for example [13]).

Theorems 1.1 and 1.2 concern the eigenspace \mathcal{E}_n of the harmonic oscillator, with n odd. When n is even, the picture is different. Some nodal sets have compact connected components, with or without critical zeros, some have both compact and non-compact components. Other examples can be analyzed as well. This will be analyzed in the future.

Acknowledgements.

The second author would like to thank D. Jakobson, I. Polterovich and M. Persson-Sundqvist for useful discussions, transmission of information or computations.

2 A reminder on Hermite polynomials

We use the definition, normalization, and notations of Szegő's book [16]. With these choices, H_n has the following properties, [16, § 5.5 and Theorem 6.32].

1. H_n satisfies the differential equation

$$y''(t) - 2t y'(t) + 2n y(t) = 0.$$

2. $H_n(t)$ is a polynomial of degree n which is even (*resp.* odd) for n even (*resp.* odd).
3. $H_n(t) = 2t H_{n-1}(t) - 2(n-1) H_{n-2}(t)$, $n \geq 2$, $H_0(t) = 1$, $H_1(t) = 2t$.
4. H_n has n simple zeros $t_{n,1} < t_{n,2} < \dots < t_{n,n}$.

- 5.

$$H_n(t) = 2t H_{n-1}(t) - H'_{n-1}(t).$$

6.

$$H'_n(t) = 2nH_{n-1}(t). \quad (2.1)$$

7. The coefficient of t^n in H_n is 2^n .

8.

$$\int_{-\infty}^{+\infty} e^{-t^2} |H_n(t)|^2 dt = \pi^{\frac{1}{2}} 2^n n!.$$

9. The first zero $t_{n,1}$ of H_n satisfies

$$t_{n,1} = (2n+1)^{\frac{1}{2}} - 6^{-\frac{1}{2}}(2n+1)^{-\frac{1}{6}}(i_1 + \epsilon_n), \quad (2.2)$$

where i_1 is the first positive real zero of the Airy function, and $\lim_{n \rightarrow +\infty} \epsilon_n = 0$.

The following result (Theorem 7.6.1 in Szegő's book [16]) will also be useful:

Lemma 2.1 *The successive relative maxima of $|H_n(t)|$ form an increasing sequence for $t \geq 0$.*

Proof.

It is enough to observe that the function

$$\Theta_n(t) := 2nH_n(t)^2 + H'_n(t)^2$$

satisfies

$$\Theta'_n(t) = 4t (H'_n(t))^2.$$

□

3 Stern-like constructions for the harmonic oscillator, case n -odd

3.1 The case of the square

Consider the square $[0, \pi]^2$, with Dirichlet boundary conditions, and the following families of eigenfunctions associated with the eigenvalues $\hat{\lambda}(1, 2r) := 1 + 4r^2$, where r is a positive integer, and $\theta \in [0, \pi/4]$,

$$(x, y) \mapsto \cos \theta \sin x \sin(2ry) + \sin \theta \sin(2rx) \sin y.$$

According to [14], for any given r , the typical evolution of the nodal sets when θ varies is similar to the case $r = 4$ shown in Figure 1 [1, Figure 6.9]: generally speaking, the nodal sets deform continuously, except for finitely many values of θ , for which crossings appear or disappear.

We would like to get similar results for the isotropic quantum harmonic oscillator.

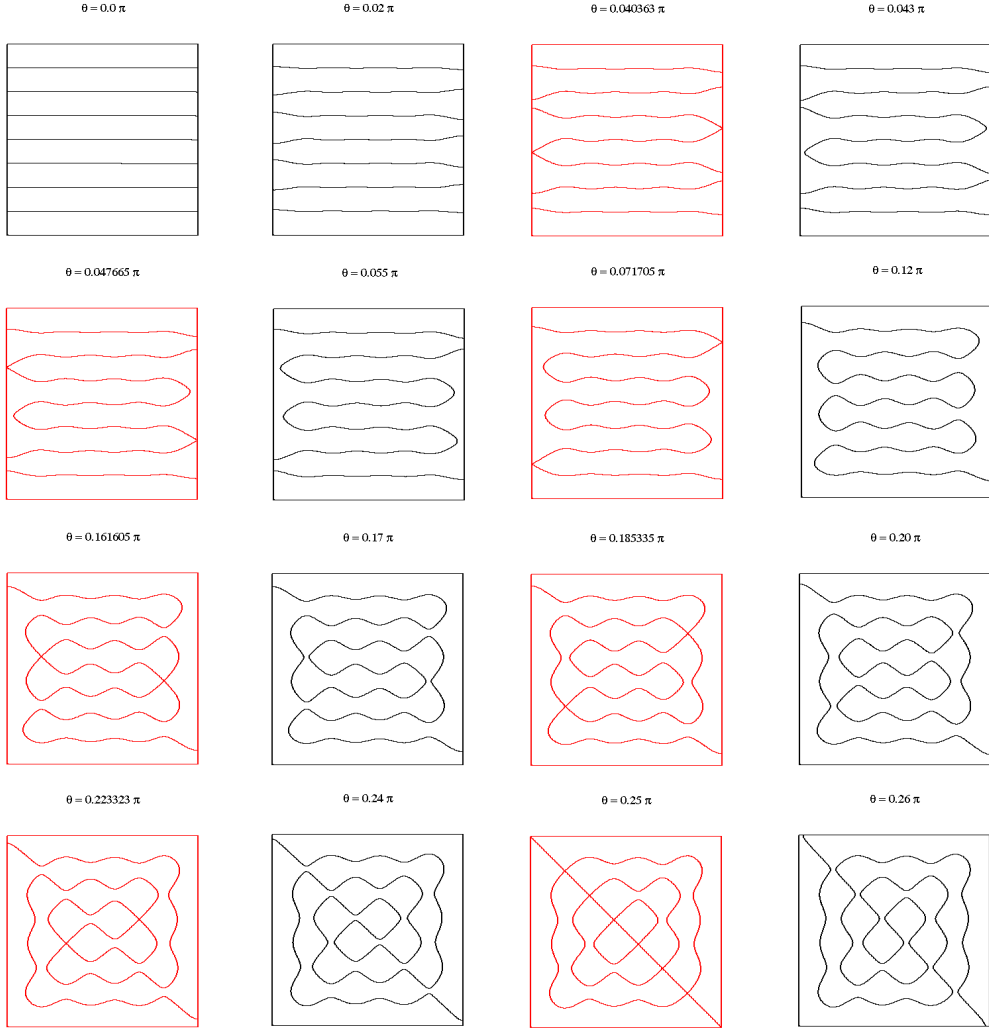


Figure 1: Evolution of the nodal set in the case of the square.

3.2 Symmetries

Recall the notation,

$$\Phi_n^\theta(x, y) := \Phi_n(x, y, \theta) := \cos \theta \phi_{n,0} + \sin \theta \phi_{0,n}. \quad (3.1)$$

Since $\Phi_n^{\theta+\pi} = -\Phi_n^\theta$, it suffices to vary the parameter θ in the interval $[0, \pi[$.

Since n is odd, we have the following symmetries.

$$\begin{cases} \Phi_n^\theta(-x, y) &= \Phi_n^{\pi-\theta}(x, y), \\ \Phi_n^\theta(x, -y) &= -\Phi_n^{\pi-\theta}(x, y), \\ \Phi_n^\theta(y, x) &= \Phi_n^{\frac{\pi}{2}-\theta}(x, y). \end{cases} \quad (3.2)$$

When n is odd, it therefore suffices to vary the parameter θ in the interval $[0, \frac{\pi}{4}]$. The

case $\theta = 0$ is particular, so that we shall mainly consider $\theta \in]0, \frac{\pi}{4}]$.

3.3 Critical zeros

A *critical zero* of Φ_n^θ is a point $(x, y) \in \mathbb{R}^2$ such that both Φ_n^θ and its differential $d\Phi_n^\theta$ vanish at (x, y) . The critical zeros of Φ_n^θ satisfy the following equations.

$$\begin{cases} \cos \theta H_n(x) + \sin \theta H_n(y) &= 0, \\ \cos \theta H'_n(x) &= 0, \\ \sin \theta H'_n(y) &= 0. \end{cases} \quad (3.3)$$

Equivalently, using the properties of the Hermite polynomials, a point (x, y) is a critical zero of Φ_n^θ if and only if

$$\begin{cases} \cos \theta H_n(x) + \sin \theta H_n(y) &= 0, \\ \cos \theta H_{n-1}(x) &= 0, \\ \sin \theta H_{n-1}(y) &= 0. \end{cases} \quad (3.4)$$

The only possible critical zeros of the eigenfunction Φ_n^θ are the points $(t_{n-1,i}, t_{n-1,j})$ for $1 \leq i, j \leq (n-1)$, where the coordinates are the zeros of the Hermite polynomial H_{n-1} . The point $(t_{n-1,i}, t_{n-1,j})$ is a critical zero of Φ_n^θ if and only if $\theta = \theta(i, j)$, where $\theta(i, j) \in]0, \pi[$ is uniquely defined by the equation,

$$\cos(\theta(i, j)) H_n(t_{n-1,i}) + \sin(\theta(i, j)) H_n(t_{n-1,j}) = 0. \quad (3.5)$$

Here we have used the fact that H_n and H'_n have no common zeros. We have proved the following lemma.

Lemma 3.1 *For $\theta \in [0, \pi[$, the eigenfunction Φ_n^θ has no critical zero, unless θ is one of the $\theta(i, j)$ defined by equation (3.5). In particular Φ_n^θ has no critical zero, except for finitely many values of the parameter $\theta \in [0, \pi[$. Let $\theta_0 = \theta(i_0, j_0)$, defined by some $(t_{n-1,i_0}, t_{n-1,j_0})$. The function $\Phi_n^{\theta_0}$ has finitely many critical zeros, namely the points $(t_{n-1,i}, t_{n-1,j})$ which satisfy*

$$\cos \theta_0 H_n(t_{n-1,i}) + \sin \theta_0 H_n(t_{n-1,j}) = 0, \quad (3.6)$$

among them the point $(t_{n-1,i_0}, t_{n-1,j_0})$.

Remarks.

From the general properties of nodal lines [1, Properties 5.2], we derive the following facts.

1. When $\theta \notin \{\theta(i, j) \mid 1 \leq i, j \leq n-1\}$, the nodal set of the eigenfunction Φ_n^θ , denoted by $N(\Phi_n^\theta)$, is a smooth 1-dimensional submanifold of \mathbb{R}^2 .

2. When $\theta \in \{\theta(i, j) \mid 1 \leq i, j \leq n-1\}$, the nodal set $N(\Phi_n^\theta)$ has finitely many singularities which are double crossings. Indeed, the Hessian of the function Φ_n^θ at a critical zero $(t_{n-1,i}, t_{n-1,j})$ is given by

$$\text{Hess}_{(t_{n-1,i}, t_{n-1,j})} \Phi_n^\theta = \exp\left(-\frac{t_{n-1,i}^2 + t_{n-1,j}^2}{2}\right) \begin{pmatrix} \cos \theta H_n''(t_{n-1,i}) & 0 \\ 0 & \sin \theta H_n''(t_{n-1,j}) \end{pmatrix},$$

and the assertion follows from the fact that H_{n-1} has simple zeros.

3.4 General properties of the nodal set $N(\Phi_n^\theta)$

Denote by \mathcal{L} the finite lattice

$$\mathcal{L} := \{(t_{n,i}, t_{n,j}) \mid 1 \leq i, j \leq n\} \subset \mathbb{R}^2, \quad (3.7)$$

consisting of points whose coordinates are the zeros of the Hermite polynomial H_n . Since we can assume that $\theta \in]0, \frac{\pi}{4}]$, we have the following inclusions for the nodal set,

$$\mathcal{L} \subset N(\Phi_n^\theta) \subset \mathcal{L} \cup \{(x, y) \in \mathbb{R}^2 \mid H_n(x) H_n(y) < 0\}. \quad (3.8)$$

Remarks. Assume that $\theta \in]0, \frac{\pi}{4}]$.

- (i) The nodal set $N(\Phi_n^\theta)$ cannot meet the vertical lines $\{x = t_{n,i}\}$, or the horizontal lines $\{y = t_{n,i}\}$ away from the set \mathcal{L} .
- (ii) The lattice point $(t_{n,i}, t_{n,j})$ is not a critical zero of Φ_n^θ (because H_n and H_n' have no common zero). As a matter of fact, near a lattice point, the nodal set $N(\Phi_n^\theta)$ is a single arc through the lattice point, with a tangent which is neither horizontal, nor vertical.

Figure 2 shows the evolution of the nodal set of Φ_n^θ when θ varies in the interval $]0, \frac{\pi}{4}]$. The values of θ with two digits are regular values (*i.e.* correspond to an eigenfunction without critical zeros), the values of θ with at least three digits are critical values (*i.e.* correspond to an eigenfunction with critical zeros). The form of the nodal set is stable between two consecutive critical values of the parameter θ . In the figures, the grey lines correspond to the zeros of H_7 . The blue lines correspond to the zeros of H_7' , *i.e.* to the zeros of H_6 .

We now describe the nodal set $N(\Phi_n^\theta)$ outside a large enough square which contains the lattice \mathcal{L} . For this purpose, we give two *barrier lemmas*.

Lemma 3.2 *Assume that $\theta \in]0, \frac{\pi}{4}]$. For $n \geq 1$, define $t_{n-1,0}$ to be the unique point in $] - \infty, t_{n,1}[$ such that $H_n(t_{n-1,0}) = -H_n(t_{n-1,1})$. Then,*

1. $\forall t \leq t_{n,1}$, the function $y \mapsto \Phi_n^\theta(t, y)$ has exactly one zero in the interval $[t_{n,n}, +\infty[$;
2. $\forall t < t_{n-1,0}$, the function $y \mapsto \Phi_n^\theta(t, y)$ has exactly one zero in the interval $] - \infty, +\infty[$.

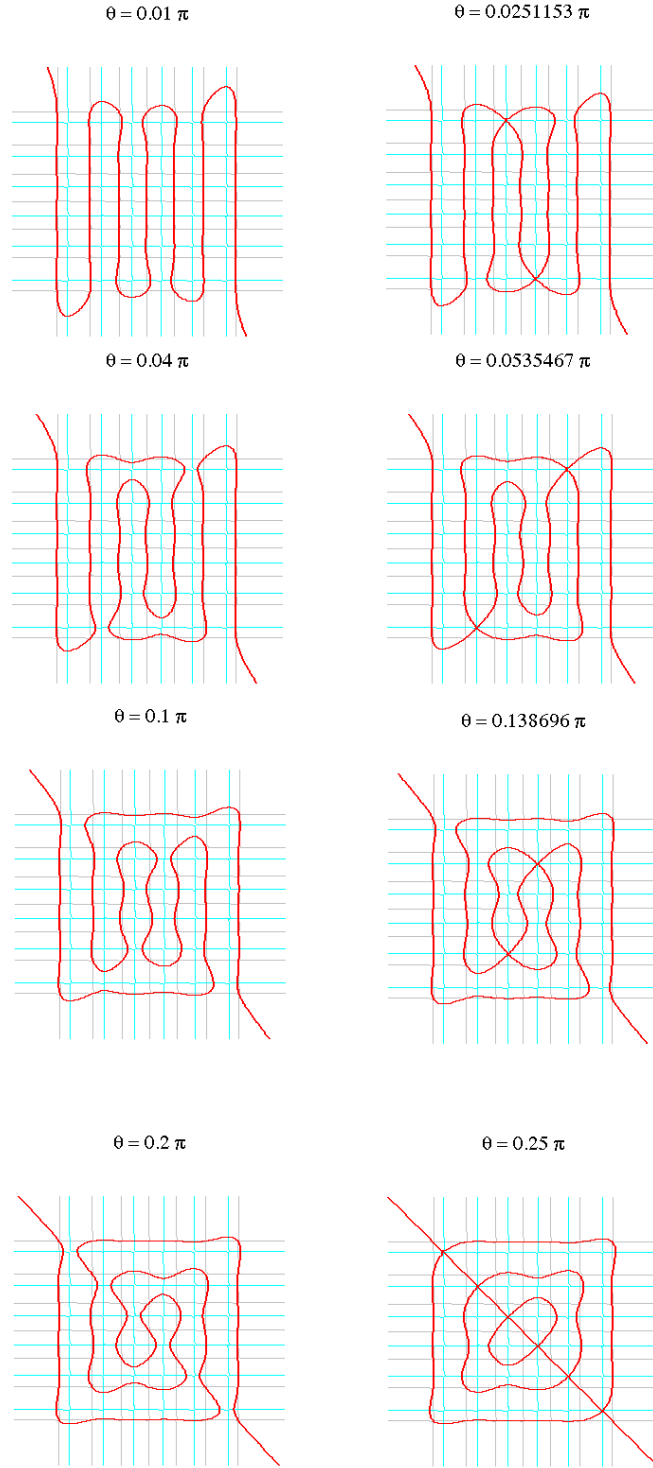


Figure 2: Evolution of the nodal set $N(\Phi_n^\theta)$, for $\theta \in]0, \frac{\pi}{4}]$.

Proof.

Let $v(y) := \exp(\frac{t^2+y^2}{2}) \Phi_n^\theta(t, y)$. In $]t_{n,n}, +\infty[$, $v'(y)$ is positive, and $v(t_{n,n}) \leq 0$. The first

assertion follows. The local extrema of v occur at the points $t_{n-1,j}$, for $1 \leq j \leq (n-1)$. The second assertion follows from the definition of $t_{n-1,0}$, and from the inequalities,

$$\begin{aligned} \cos \theta H_n(t) + \sin \theta H_n(t_{n-1,j}) &\leq \frac{1}{\sqrt{2}} \left(H_n(t) - |H_n|(t_{n-1,j}) \right) \\ &< -\frac{1}{\sqrt{2}} \left(H_n(t_{n-1,1}) - |H_n|(t_{n-1,j}) \right) \leq 0, \end{aligned}$$

where we have used Lemma 2.1. □

Remark. Using the symmetry with respect to the vertical line $\{x = 0\}$, one has similar statements for $t \geq t_{n,n}$ and for $t > -t_{n-1,0}$.

Lemma 3.3 *Let $\theta \in]0, \frac{\pi}{4}]$. Define $t_{n-1,n}^\theta \in]t_{n,n}, \infty[$ to be the unique point such that $\tan \theta H_n(t_{n-1,n}^\theta) = H_n(t_{n-1,1})$. Then,*

1. $\forall t \geq t_{n,n}$, the function $x \mapsto \Phi_n^\theta(x, t)$ has exactly one zero in the interval $] -\infty, t_{n,1}]$;
2. $\forall t > t_{n-1,n}^\theta$, the function $x \mapsto \Phi_n^\theta(x, t)$ has exactly one zero in the interval $] -\infty, \infty[$.
3. For $\theta_2 > \theta_1$, we have $t_{n-1,n}^{\theta_2} < t_{n-1,n}^{\theta_1}$.

Proof. Let $h(x) := \exp(\frac{x^2+t^2}{2}) \Phi_n^\theta(x, t)$. In the interval $] -\infty, t_{n,1}]$, the derivative $h'(x)$ is positive, $h(t_{n,1}) > 0$, and $\lim_{x \rightarrow -\infty} h(x) = -\infty$, since n is odd. The first assertion follows. The local extrema of h are achieved at the points $t_{n-1,j}$. Using Lemma 2.1, for $t \geq t_{n-1,n}^\theta$, we have the inequalities,

$$\begin{aligned} H_n(t_{n-1,j}) + \tan \theta H_n(t) &\geq \tan \theta H_n(t_{n-1,n}^\theta) - |H_n(t_{n-1,j})| \\ &= H_n(t_{n-1,1}) - |H_n(t_{n-1,j})| \geq 0. \end{aligned}$$

□

Remark. Using the symmetry with respect to the horizontal line $\{y = 0\}$, one has similar statements for $t \leq t_{n,1}$ and for $t < -t_{n-1,n}^\theta$.

As a consequence of the above lemmas, we have the following description of the nodal set far enough from $(0, 0)$.

Proposition 3.4 *Let $\theta \in]0, \frac{\pi}{4}]$. In the set $\mathbb{R}^2 \setminus]-t_{n-1,n}^\theta, t_{n-1,n}^\theta[\times]t_{n-1,0}, |t_{n-1,0}||$, the nodal set $N(\Phi_n^\theta)$ consists of two regular arcs. The first arc is a graph $y(x)$ over the interval $] -\infty, t_{n,1}]$, starting from the point $(t_{n,1}, t_{n,n})$ and escaping to infinity with,*

$$\lim_{x \rightarrow -\infty} \frac{y(x)}{x} = -\sqrt[n]{\cot \theta}.$$

The second arc is the image of the first one under the symmetry with respect to $(0, 0)$ in \mathbb{R}^2 .

3.5 Local nodal patterns

As in the case of the square, we study the possible local nodal patterns taking into account the fact that the nodal set contains the lattice points \mathcal{L} , can only intersect the connected components of the set $\{H_n(x) H_n(y) < 0\}$, and consists of a simple arc at the lattice points. The following figure summarized the possible nodal patterns in the interior of the square [1, Figure 6.4],

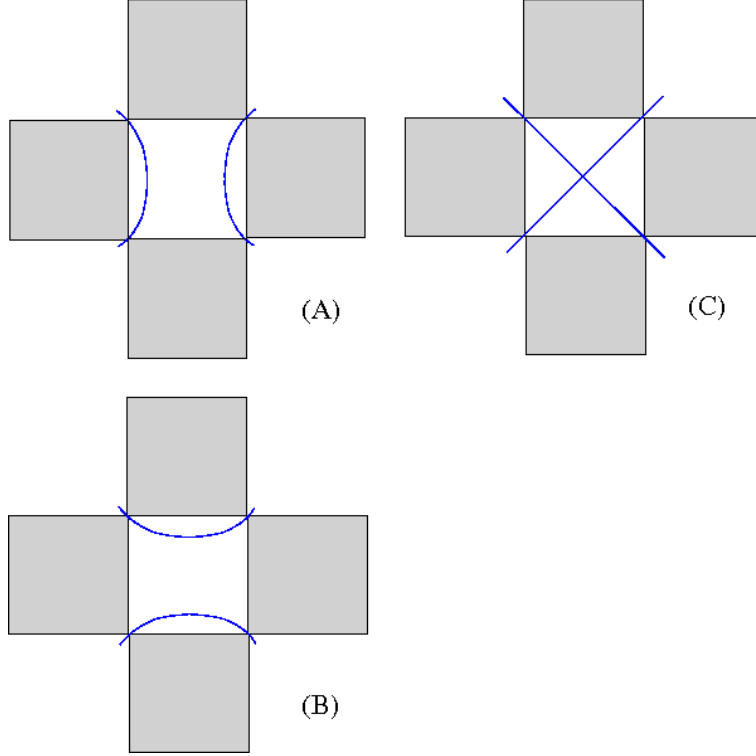


Figure 3: Local nodal patterns.

Except for nodal arcs which escape to infinity, the local nodal patterns for the quantum harmonic oscillator are the same (note that in the present case, the connected components of the set $\{H_n(x) H_n(y) < 0\}$ are rectangles, no longer equal squares).

Case (C) occurs near a critical zero. Following the same ideas as in the case of the square, in order to decide between cases (A) and (B), we use the barrier lemmas, Lemma 3.2 or 3.3, the vertical lines $\{x = t_{n-1,j}\}$, or the horizontal lines $\{y = t_{n-1,j}\}$.

4 Proof of Theorem 1.1

Note that

$$\phi_{n,0}(x, y) - \phi_{0,n}(x, y) = -\Phi_n^{\frac{3\pi}{4}}(x, y) = -\Phi_n^{\frac{\pi}{4}}(y, x).$$

Hence it is the same to work with $\theta = \frac{\pi}{4}$ and the anti-diagonal, or to work with $\theta = \frac{3\pi}{4}$ and the diagonal. From now on, we work near $\frac{3\pi}{4}$.

4.1 The nodal set of $\Phi_n^{\frac{3\pi}{4}}$

Proposition 4.1 *Let $\{t_{n-1,i}, 1 \leq i \leq n-1\}$ denote the zeroes of H_{n-1} . For n odd, the nodal set of $\phi_{n,0} - \phi_{0,n}$ consists of the diagonal $x = y$, and of $\frac{n-1}{2}$ disjoint simple closed curves crossing the diagonal at the $(n-1)$ points $(t_{n-1,i}, t_{n-1,i})$, and the antidiagonal at the $(n-1)$ points $(t_{n,i}, -t_{n,i})$.*

To prove Proposition 4.1, we first observe that it is enough to analyze the zero set of

$$(x, y) \mapsto \Psi_n(x, y) := H_n(x) - H_n(y).$$

The critical points of Ψ_n are determined by

$$H'_n(x) = 0, H'_n(y) = 0.$$

Hence, the critical points of Ψ_n consist of the $(n-1)^2$ points $(t_{n-1,i}, t_{n-1,j})$, for $1 \leq i, j \leq (n-1)$, where $t_{n-1,i}$ is the i -th zero of the polynomial H_{n-1} .

The zero set of Ψ_n contains the diagonal $\{x = y\}$. Since n is odd, there are only n points belonging to the zero set on the anti-diagonal $\{x + y = 0\}$.

On the diagonal, there are $(n-1)$ critical points. We claim that there are no critical zeros outside the diagonal. Indeed, let $(t_{n-1,i}, t_{n-1,j})$ be a critical zero. Then, $H_n(t_{n-1,i}) = H_n(t_{n-1,j})$. Using Lemma 2.1 and the parity properties of Hermite polynomials, we see that $|H_n(t_{n-1,i})| = |H_n(t_{n-1,j})|$ occurs if and only if $t_{n-1,i} = \pm t_{n-1,j}$. Since n is odd, we can conclude that $H_n(t_{n-1,i}) = H_n(t_{n-1,j})$ occurs if and only if $t_{n-1,i} = t_{n-1,j}$.

4.2 Existence of disjoint simple closed curves in the nodal set of $\Phi_n^{\frac{3\pi}{4}}$

The second part in the proof of the proposition follows closely the proof in the case of the square (see Section 5 in [1]). Essentially, the Chebyshev polynomials are replaced by the Hermite polynomials. Note however that the checkerboard is no more with equal squares, and that the square $[0, \pi]^2$ has to be replaced in the argument by the rectangle $[t_{n-1,0}, -t_{n-1,0}] \times [-t_{n-1,n}^\theta, t_{n-1,n}^\theta]$, for some θ such that $0 < \theta < \frac{3\pi}{4}$, see Lemmas 3.2 and 3.3.

The checkerboard argument holds, see (3.8).

The separation lemmas of our previous paper [1] must be substituted by Lemmas 3.2 and 3.3, and similar statements with the lines $\{x = t_{n-1,j}\}$ and $\{y = t_{n-1,j}\}$, for $1 \leq j \leq (n-1)$.

One needs to control what is going on at infinity. As a matter of fact, outside a specific rectangle centered at the origin, the zero set is the diagonal $\{x = y\}$, see Proposition 3.4.

Hence in this way (like for the square), we get that inside the zero set, we have the diagonal and $\frac{n-1}{2}$ disjoint simple closed lines turning around the origin.

4.3 No other closed curve in the nodal set of $\Phi_n^{\frac{3\pi}{4}}$

It remains to show that there are no other closed curves which do not cross the diagonal. The “energy” considerations of our previous papers work in the following way.

Assume there is a connected component of the nodal set which does not meet the lattice \mathcal{L} . Using Proposition 3.4, we see that this component must be contained in some large coordinate square centered at $(0,0)$, call it C . Since the nodal set cannot meet the vertical or horizontal lines defined by the zeros of H_n , we would have a nodal domain ω contained in C , hence in one of the bounded connected components of $\{H_n(x)H_n(y) < 0\}$, and hence also in some infinite rectangle R between two consecutive zeros of H_n . We can compute the energy for ω by applying Green’s formula in ω . We can compute the energy of the infinite rectangle R by applying Green’s formula first in a finite rectangle, and then taking the limit (using the decaying exponential factor). We have that the first Dirichlet eigenvalues λ_1 satisfy $\lambda_1(\omega) = \lambda_1(R) = 2(n+1)$. On the other hand, taking some ω_1 such that $\omega \subset \omega_1 \subset R$, with strict inclusions, we have $\lambda_1(\omega) > \lambda_1(\omega_1) \geq \lambda_1(R)$, a contradiction.

A simple alternative argument is the following. We look at the line $y = \alpha x$ for some $\alpha \neq 1$. The intersection of the zero set with this line corresponds to the zeroes of the polynomial $x \mapsto H_n(x) - H_n(\alpha x)$ which has at most n zeroes. But in our previous construction, we get at least n zeroes. So the presence of extra curves would lead to a contradiction for some α . This argument solves the problem at infinity as well.

4.4 Perturbation argument

Figure 4 shows the desingularization of the nodal set $N(\Phi_n^{\frac{3\pi}{4}})$, from below and from above. The picture is the same as in the case of the square (see Figure 1), all the critical points disappear at the same time and in the same manner, *i.e.* all the double crossings open up horizontally or vertically depending whether θ is less than or bigger than $\frac{3\pi}{4}$.

As in the case of the square, in order to show that the nodal set can be desingularized under small perturbation, we look at the signs of the eigenfunction $\Phi_n^{\frac{3\pi}{4}}$ near the critical zeros. We use the cases [i] and [ii] which appear in Figure 5 below (see also [1, Figure 6.7]).

The sign configuration for $\phi_{n,0}(x,y) - \phi_{0,n}(x,y)$ near the critical zero $(t_{n-1,i}, t_{n-1,i})$ is that

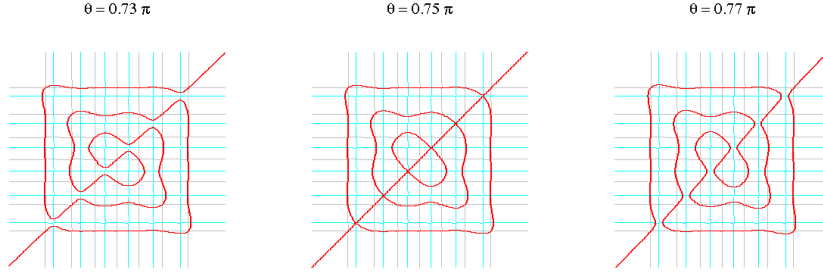


Figure 4: The nodal set of $N(\Phi_n^\theta)$ near $\frac{3\pi}{4}$.

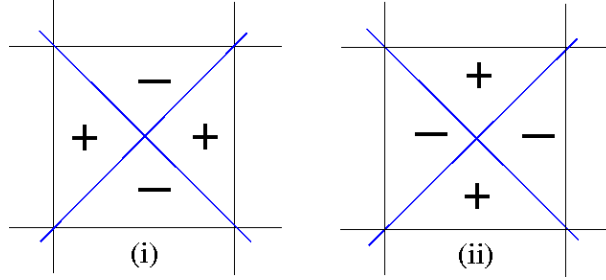


Figure 5: Signs near the critical zeros.

of

$$\begin{cases} \text{case [i],} & \text{if } i \text{ is even,} \\ \text{case [ii],} & \text{if } i \text{ is odd.} \end{cases}$$

Looking at the intersection of the nodal set with the vertical line $\{y = t_{n-1,i}\}$, we have that

$$(-1)^i (H_n(t) - H_n(t_{n-1,i})) \geq 0, \text{ for } t \in]t_{n,i}, t_{n,i+1}[.$$

For positive ϵ small, we write

$$(-1)^i (H_n(t) - (1 + \epsilon)H_n(t_{n-1,i})) = (-1)^i (H_n(t) - H_n(t_{n-1,i})) + \epsilon(-1)^{i+1} H_n(t_{n-1,i}),$$

so that

$$(-1)^i (H_n(t) - (1 + \epsilon)H_n(t_{n-1,i})) \geq 0, \text{ for } t \in]t_{n,i}, t_{n,i+1}[.$$

A similar statement can be written for horizontal line $\{x = t_{n-1,i}\}$ and $-\epsilon$, with $\epsilon > 0$, small enough. These inequalities describe how the crossings open up all at the same time, and in the same manner, vertically (case I) or horizontally (case II), see Figure 6, as in the case of the square [1, Figure 6.8].

We can then conclude as in the case of the square, using the local nodal patterns, Section 3.5.

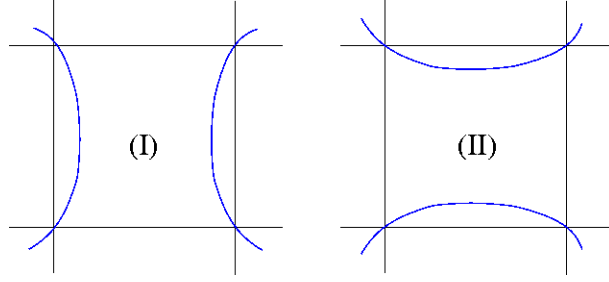


Figure 6: Desingularization at a critical zero.

Remark. Because the local nodal patterns can only change when θ passes through one of the values $\theta(i, j)$ defined in (3.5), the above arguments work for $\theta \in J \setminus \{\frac{3\pi}{4}\}$, for any interval J containing $\frac{3\pi}{4}$ and no other value $\theta(i, j)$.

5 Proof of Theorem 1.2

Proposition 5.1 *The conclusion of Theorem 1.2 holds with*

$$\theta_c := \inf \{ \theta(i, j) \mid 1 \leq i, j \leq n-1 \} . \quad (5.1)$$

Proof. The proof consists in the following steps. For simplicity, we call N the nodal set $N(\Phi_n^\theta)$.

- Step 1. By Proposition 3.4, the structure of the nodal set N is known outside a large coordinate rectangle centered at $(0, 0)$ whose sides are defined by the ad hoc numbers in Lemmas 3.2 and 3.3. Notice that the sides of the rectangle serve as barriers for the arguments using the local nodal patterns as in our paper for the square.
- Step 2. For $1 \leq j \leq n-1$, the line $\{x = t_{n-1,j}\}$ intersects the set N at exactly one point $(t_{n-1,j}, y_j)$, with $y_j > t_{n,n}$ when j is odd, *resp.* with $y_j < t_{n,1}$ when j is even. The proof is given below, and is similar to the proofs of Lemmas 3.2 or 3.3.
- Step 3. Any connected component of N has at least one point in common with the set \mathcal{L} . This follows from the argument with $y = \alpha x$ or from the energy argument (see Subsection 4.3).
- Step 4. Follow the nodal set from the point $(t_{n,1}, t_{n,n})$ to the point $(t_{n,n}, t_{n,1})$, using the analysis of the local nodal patterns as in the case of the square.

Proof of Step 2. For $1 \leq j \leq (n-1)$, define the function v_j by

$$v_j(y) := \cos \theta H_n(t_{n-1,j}) + \sin \theta H_n(y) .$$

The local extrema of v_j are achieved at the points $t_{n-1,i}$, for $1 \leq i \leq (n-1)$, and we have

$$v_j(t_{n-1,i}) = \cos \theta H_n(t_{n-1,j}) + \sin \theta H_n(t_{n-1,i}),$$

which can be rewritten, using (3.5), as

$$v_j(t_{n-1,i}) = \frac{H_n(t_{n-1,j})}{\sin \theta(j,i)} \sin(\theta(j,i) - \theta).$$

The first term in the right-hand side has the sign of $(-1)^{j+1}$ and the second term is positive provided that $0 < \theta < \theta_c$. Under this last assumption, we have

$$(-1)^{j+1} v_j(t_{n-1,i}) > 0, \quad \forall i, \quad 1 \leq i \leq (n-1). \quad (5.2)$$

The assertion follows. \square

6 Courant's theorem for the $2D$ quantum harmonic oscillator

Recall that \mathcal{E}_ℓ is the eigenspace of \hat{H} associated with the eigenvalue $\hat{\lambda}(\ell) := 2(\ell + 1)$. This eigenspace is generated by the eigenfunctions $\phi_{\ell-j,j}$, for $0 \leq j \leq \ell$. It has dimension $(\ell + 1)$. The functions in \mathcal{E}_ℓ are even (*resp.* odd) under the map $a : (x, y) \mapsto (-x, -y)$ when ℓ is even (*resp.* odd).

Since $\dim(\bigoplus_{j=0}^{\ell-1} \mathcal{E}_j) = \frac{\ell(\ell+1)}{2}$, Courant's theorem gives the following estimate for the number $\mu(u)$ of nodal domains of an eigenfunction $u \in \mathcal{E}_\ell$,

$$\mu(u) \leq \frac{\ell(\ell+1)}{2} + 1 =: \mu_C(\ell). \quad (6.1)$$

Using the symmetry or anti-symmetry with respect to a , one can improve Courant's estimate.

Proposition 6.1 *Let $u \in \mathcal{E}_\ell$. Then, the number $\mu(u)$ of nodal domains of u satisfies the inequalities,*

$$\mu(u) \leq \mu_L(\ell) := \begin{cases} 2(r^2 + 1) & \text{if } \ell = 2r, \\ 2r(r+1) + 2 & \text{if } \ell = 2r + 1. \end{cases} \quad (6.2)$$

In particular, we have that $\mu_L(\ell) < \mu_C(\ell)$ provided that $\ell \geq 3$, and $\mu_L(\ell) = \mu_C(\ell)$, when $\ell = 2$.

Corollary 6.2 *The only Courant sharp eigenvalues of the quantum harmonic oscillator are the eigenvalues,*

$$\begin{cases} \hat{\lambda}(0) = 2, & \text{with } \mu_C(0) = 1, \\ \hat{\lambda}(1) = 4, & \text{with } \mu_C(1) = 2, \\ \hat{\lambda}(2) = 6, & \text{with } \mu_C(2) = 4. \end{cases} \quad (6.3)$$

Proof of the corollary. The first two assertions are clear. The last one follows from the fact that the nodal set of an eigenfunction in \mathcal{E}_2 is a hyperbola, the union of two lines which intersect, or an ellipse. \square

Proof of the proposition. We use Leydold's argument in [10], namely the symmetry properties of the eigenfunctions with respect to a , the fact that an odd eigenfunction is always orthogonal to an even one, and Courant's proof.

- Assume that $u \in \mathcal{E}_\ell$ with $\ell = 2r$. We have

$$\dim\left(\bigoplus_{j=0}^{r-1} \mathcal{E}_{2j}\right) = r^2.$$

There are k_i nodal domains of u which are a invariant, $a(\omega) = \omega$, and $2k_a$ nodal domains which are not invariant, $a(\omega) \cap \omega = \emptyset$. Assume that $k_i + k_a \geq r^2 + 2$. Define functions u_j such that $u_j = u|_{\omega_j}$, and 0 elsewhere, for each invariant domain ω_j , $1 \leq j \leq k_i - 1$, and $u_j = u|_{\omega_p \cup a(\omega_p)}$, and 0 elsewhere, for the k_a non-invariant domains. This gives us $k_a + k_i - 1 \geq r^2 + 1$ independent functions. We can find a linear combination v of these functions such that $\|v\|_{L^2} = 1$, $v \perp \bigoplus_{j=0}^{r-1} \mathcal{E}_{2j}$, and $\mathcal{Q}(v) = 2(\ell + 1)$, where \mathcal{Q} is the quadratic form associated with \hat{H} . The function v is even by construction so that it is orthogonal to any odd eigenfunction. It follows that $v \in \mathcal{E}_\ell$ which leads to a contradiction since v vanishes on an open set. It follows that $k_i + k_a \leq r^2 + 1$ and hence that $k_i + 2k_a \leq 2(r^2 + 1)$. This proves the first assertion.

- Assume that $u \in \mathcal{E}_\ell$ with $\ell = 2r + 1$. The proof is similar. We have that

$$\dim\left(\bigoplus_{j=0}^{r-1} \mathcal{E}_{2j+1}\right) = r(r + 1).$$

For an odd eigenfunction u , the nodal domains satisfy $a(\omega) \cap \omega = \emptyset$, so that $\mu(u) = 2k$, and we can construct k linearly independent functions $u_j = u|_{a(\omega) \cup \omega}$, and we can proceed as above. \square

Remark. In the above proof, we used Courant's proof which is based on energy estimates, using Green's formula for the eigenfunction u . That this can be done in the case of the quantum harmonic operator follows from the following argument. At infinity, the nodal set of u is a regular submanifold. It consists of arcs asymptotic to lines determined by the homogeneous higher order terms in $\exp(\frac{x^2+y^2}{2})u(x, y)$. We can apply Green's formula to the intersections of the nodal domains of u with balls $B(0, r)$. When r tends to infinity, the boundary terms involving the ball tend to zero due to the presence of the exponential factors.

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